Division of Strength of Materials and Structures

Faculty of Power and Aeronautical Engineering

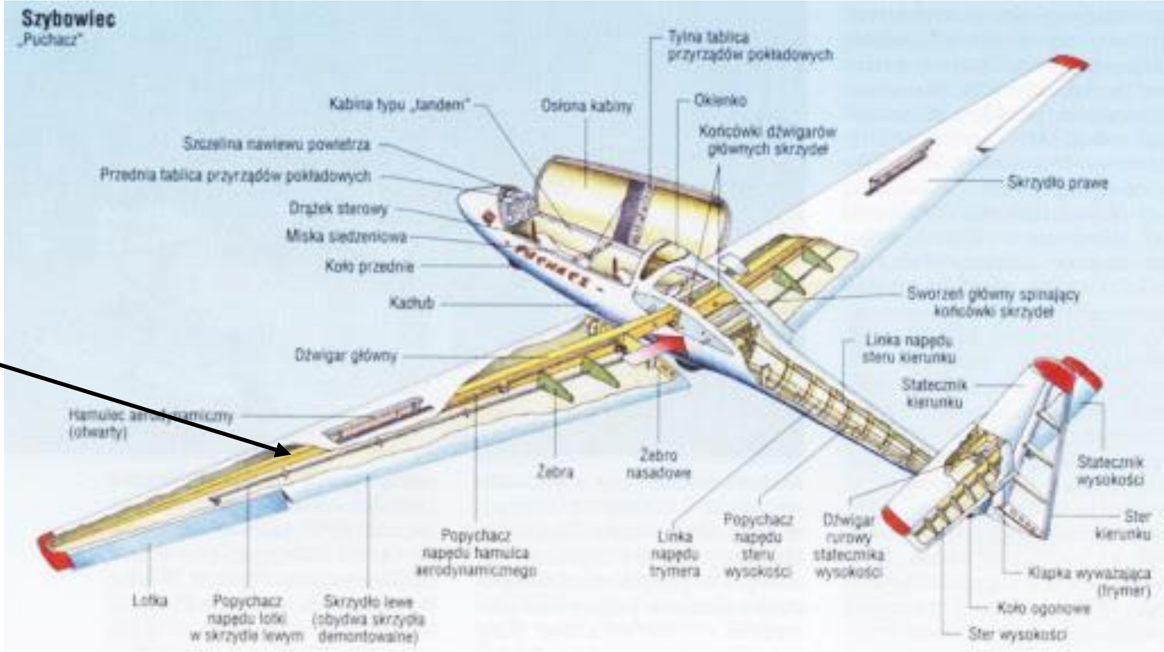
Finite element method (FEM1)

Lecture 9A. 1D beam element

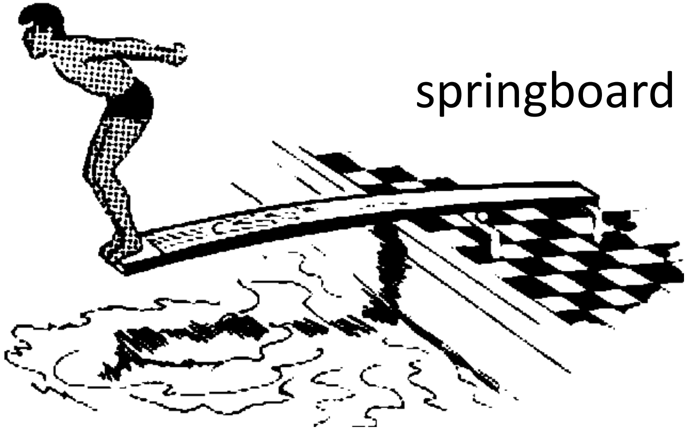
05.2026

Examples of beams

wing spar

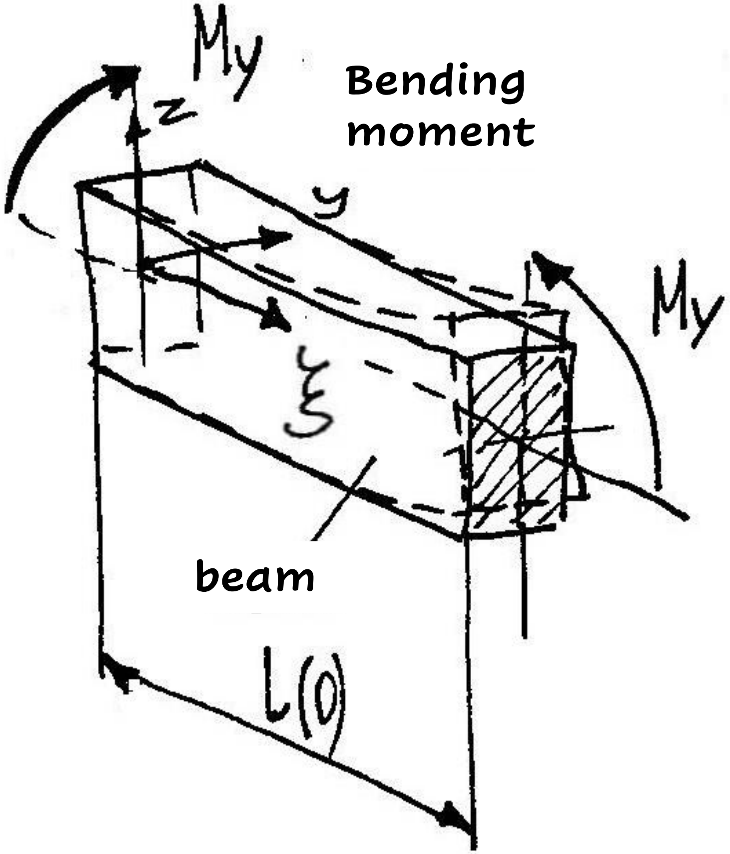


footbridge



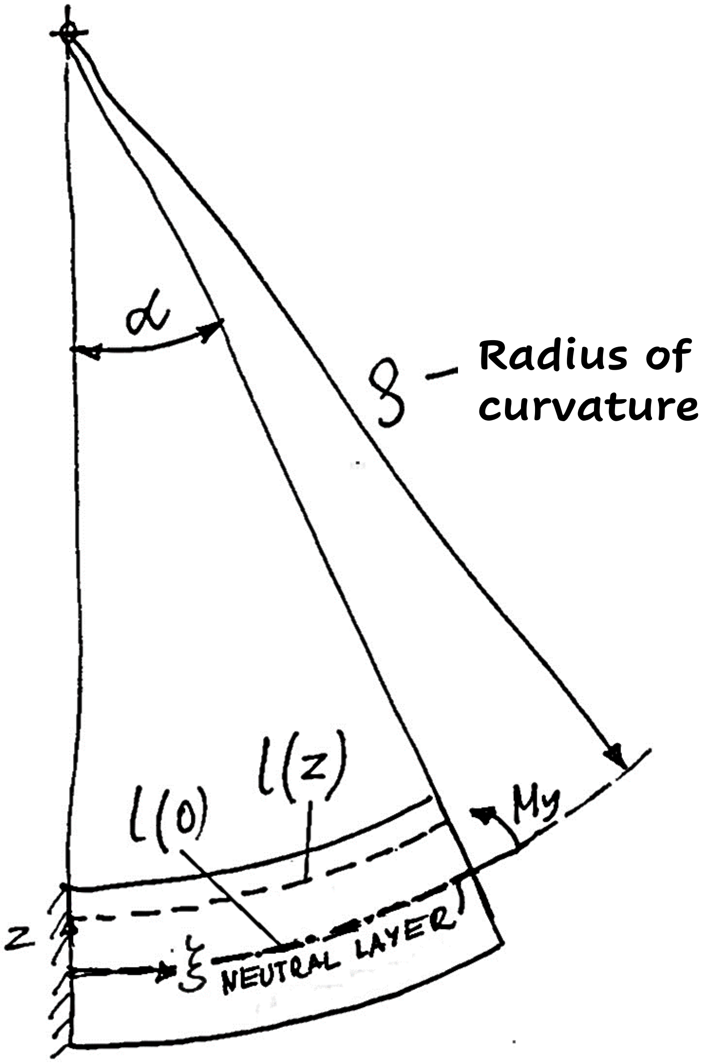
springboard

Bending without shear force (pure bending)



Bending moment

beam



ϱ – Radius of curvature

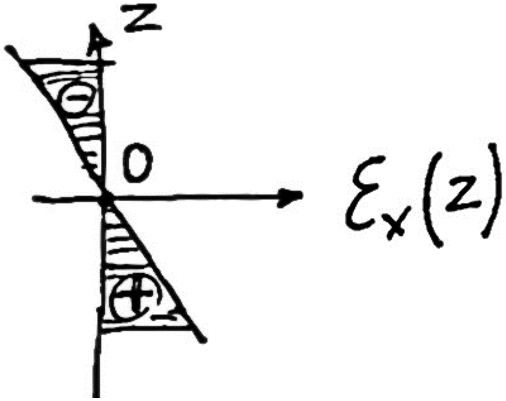
$$\varepsilon_x(z)=\frac{l(z)-l(0)}{l(0)}=\frac{\alpha(\varrho-z)-\alpha\varrho}{\alpha\varrho}=-\frac{z}{\varrho}$$

Curvature:

$$\varkappa = \frac{1}{\rho} \simeq \frac{d^2w}{d\xi^2} = w''$$

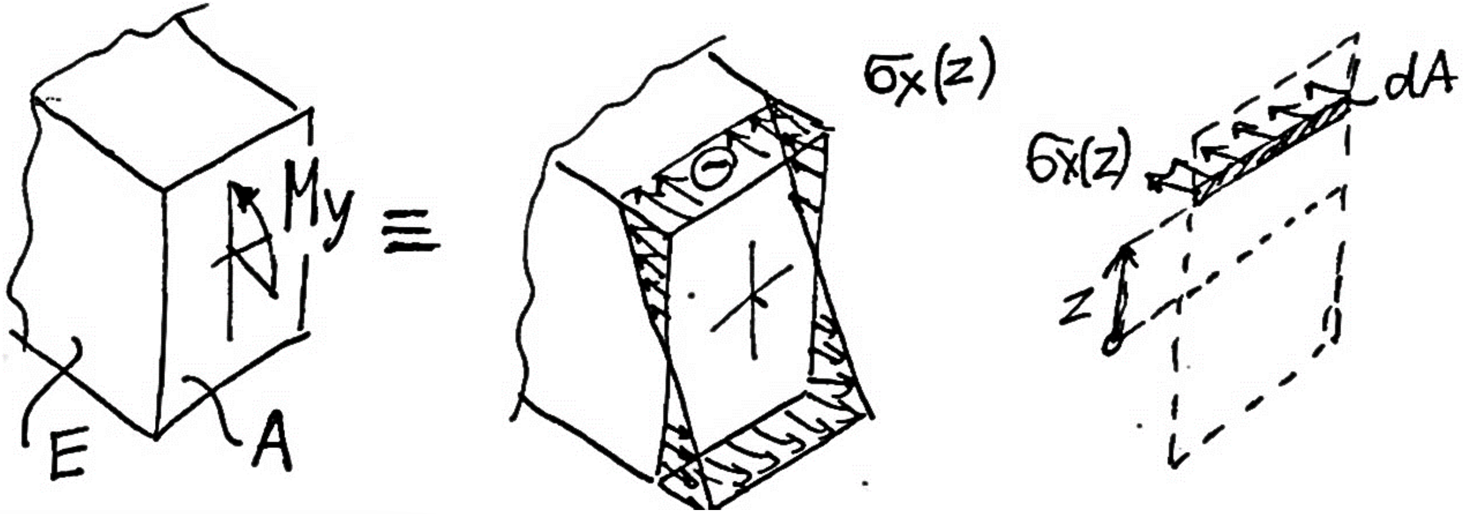
Strain:

$$\varepsilon_x(z) = -z \cdot w''$$



Stress:

$$\sigma x(z) = E \cdot \varepsilon x(z) = -E z \cdot w''$$



$$M_y = -\int_A \sigma_x(z)\cdot z\,dA = -\int_A -E z w''\cdot z\,dA =$$

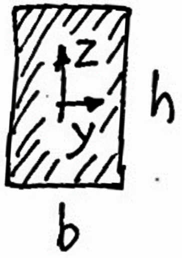
$$= E w'' \underbrace{\int_A z^2 dA}_{J_y} \Rightarrow \boxed{M_y = E J_y w''}$$

Bending moment in a beam

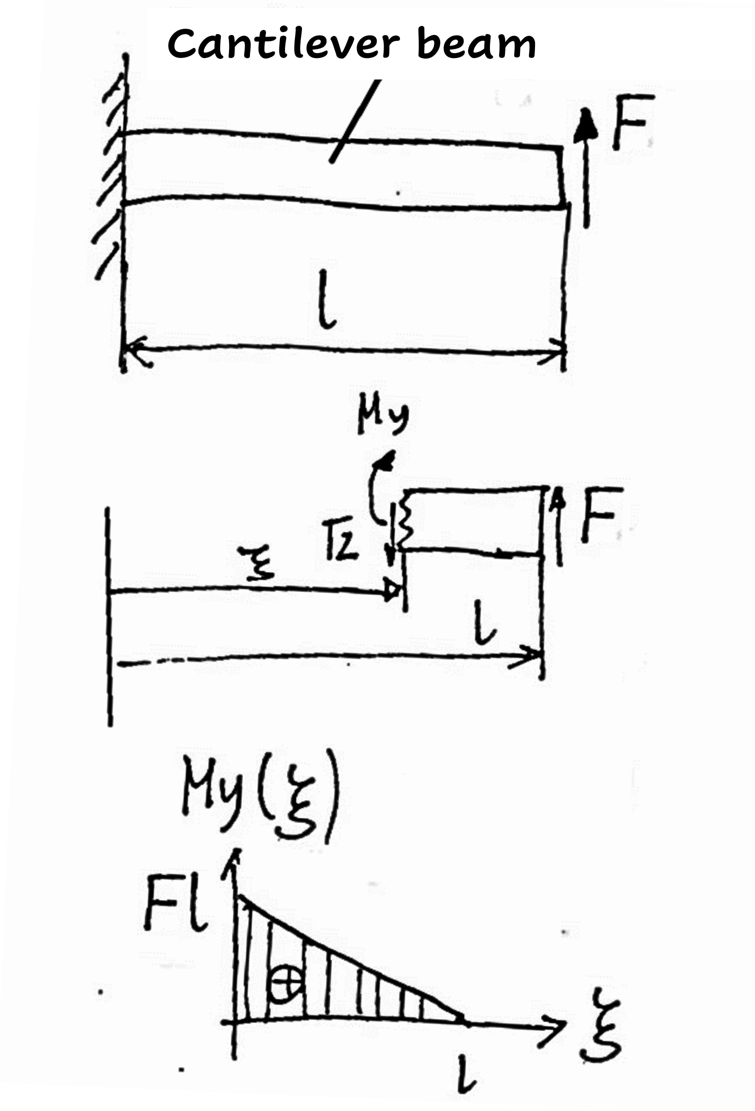
Second moment of area

$$J_y = \int_A z^2 dA = \frac{bh^3}{12}$$

For a rectangle:



Bending with shear force *(transverse bending)*



Cantilever beam

$$T_z = F$$

$$M_y - F(l - \xi) = 0$$

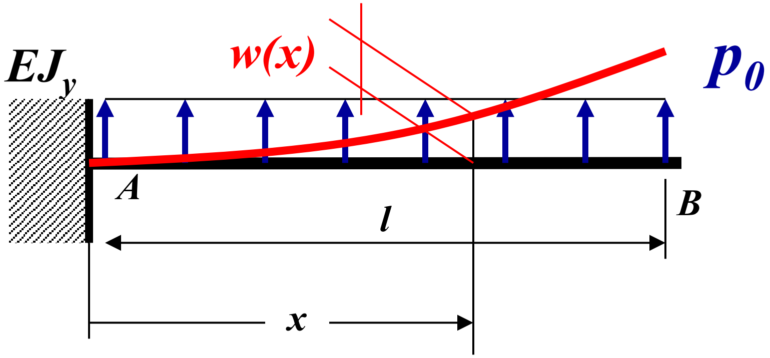
$$M_y = T_z(l - \xi)$$

$$\frac{dM_y}{d\xi} = -T_z$$

Shear force in a beam

$$\boxed{T_z = -EJ_y w'''}$$

A reminder: cantilever beam - Ritz method solution



Solve a cantilever beam using the Ritz method using a given approximation function:

$$\tilde{w}(x) = a_1 + a_2\cdot x + a_3\cdot x^2 + a_4\cdot x^3$$

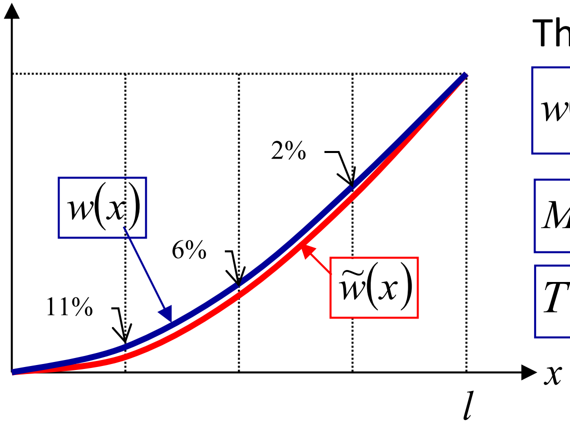
Boundary conditions: $\tilde{w}(x=0)=0 \longrightarrow a_1 = 0$ $\qquad$ $\tilde{w}'(x=0)=0 \longrightarrow a_2 = 0$

Approximate solution:

$$\tilde{w}(x) = \tfrac{5}{24}\frac{p_0 l^2}{EJ_y}\cdot x^2 - \tfrac{1}{12}\frac{p_0 l}{EJ_y}\cdot x^3$$

$$\tilde{M}_g(x) = \tfrac{5}{12}p_0 l^2 - \tfrac{1}{2}p_0 l\cdot x$$

$$\tilde{T}(x) = -\tfrac{1}{2}p_0 l$$

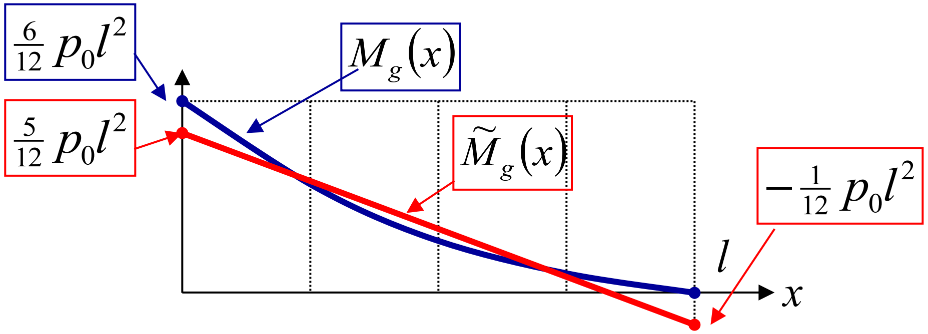


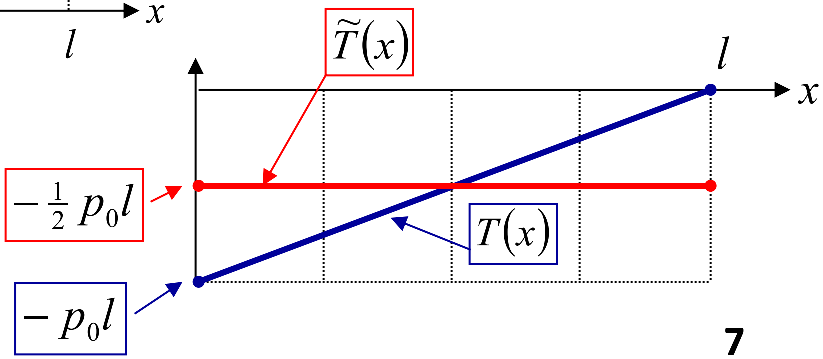
The exact solution:

$$w(x) = \tfrac{6}{24}\frac{p_0 l^2}{EJ_y}\cdot x^2 - \tfrac{2}{12}\frac{p_0 l}{EJ_y}\cdot x^3 + \tfrac{1}{24}\frac{p_0}{EJ_y}\cdot x^4$$

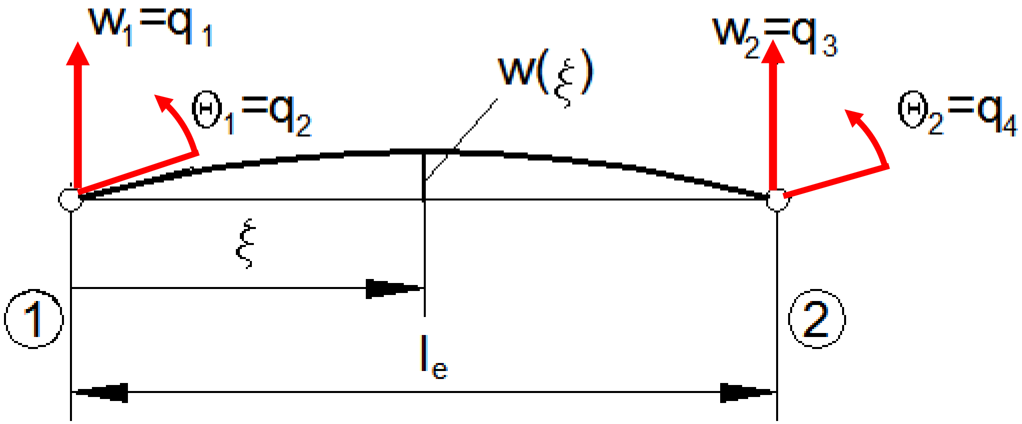
$$M_g(x) = \tfrac{1}{2}p_0(l-x)^2$$

$$T(x) = -p_0(l-x)$$





A beam finite element (bending in one plane)



q_1, q_3 – transverse displacements at nodes
q_2, q_4 – deflection angles at nodes
(positive signs in counterclockwise direction)

$n = 2 \;\; ; \;\; n_p = 2 \;\; \rightarrow \;\; n_e = n \cdot n_p = 4$

Let us assume an approximation of the deflection function in the element:

$$w(\xi) = \alpha_1 + \alpha_2 \xi + \alpha_3 \xi^2 + \alpha_4 \xi^3$$

However, new parameters are required: $w_1, w_2, \theta_1 \, , \theta_2$

Vector of nodal parameters:

$$\{q\}_e = \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}_e = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_e$$

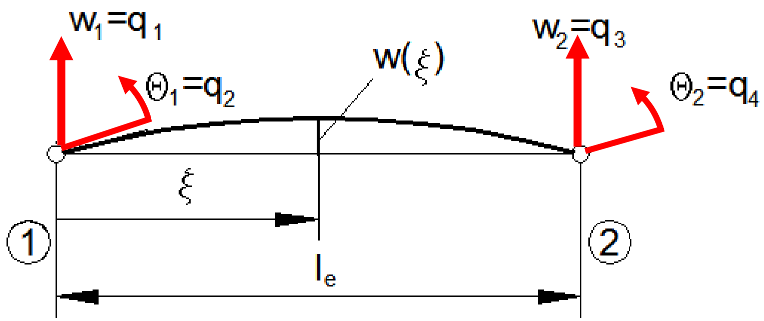
Nodal approximation:

$$w(\xi) = \sum_{i=1}^{4} N_i(\xi) q_i$$

$$w(\xi) = \lfloor N(\xi) \rfloor \{q\}_e \, ,$$

A beam finite element - the relationship between $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and q_1, q_2, q_3, q_4

$$w(\xi) = \alpha_1 + \alpha_2 \xi + \alpha_3 \xi^2 + \alpha_4 \xi^3$$



displacement at node 1 ➔ $q_1 = w(0) = \alpha_1,$

deflection angle at node 1 ➔ $q_2 = \dfrac{dw}{d\xi}(0) = \alpha_2,$

displacement at node 2 ➔ $q_3 = w(l) = \alpha_1 + \alpha_2 l_e + \alpha_3 l_e^2 + \alpha_4 l_e^3,$

deflection angle at node 2 ➔ $q_4 = \dfrac{dw}{d\xi}(l) = \alpha_2 + 2\alpha_3 l_e + 3\alpha_4 l_e^2.$

In matrix notation:

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \left[\begin{array}{c|c|c|c} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline 1 & l_e & l_e^2 & l_e^3 \\ \hline 0 & 1 & 2l_e & 3l_e^2 \end{array}\right] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}. \quad \Longrightarrow$$

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \left[\begin{array}{c|c|c|c} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline \dfrac{-3}{l_e^2} & \dfrac{-2}{l} & \dfrac{3}{l_e^2} & \dfrac{-1}{l_e} \\ \hline \dfrac{2}{l_e^3} & \dfrac{1}{l_e} & \dfrac{-2}{l_e^3} & \dfrac{1}{l_e^2} \end{array}\right] \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_e$$

A beam finite element – shape functions

$$\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{Bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ \frac{-3}{l_e^2} & \frac{-2}{l} & \frac{3}{l_e^2} & \frac{-1}{l_e}\\ \frac{2}{l_e^3} & \frac{1}{l_e} & \frac{-2}{l_e^3} & \frac{1}{l_e^2}\end{bmatrix}\begin{Bmatrix}q_1\\ q_2\\ q_3\\ q_4\end{Bmatrix}_e$$

The approximated displacement can be represented in the form:

$$w(\xi)=\lfloor 1,\xi,\xi^2,\xi^3\rfloor\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{Bmatrix}=\lfloor N_1(\xi),N_2(\xi),N_3(\xi),N_4(\xi)\rfloor\begin{Bmatrix}q_1\\ q_2\\ q_3\\ q_4\end{Bmatrix}$$

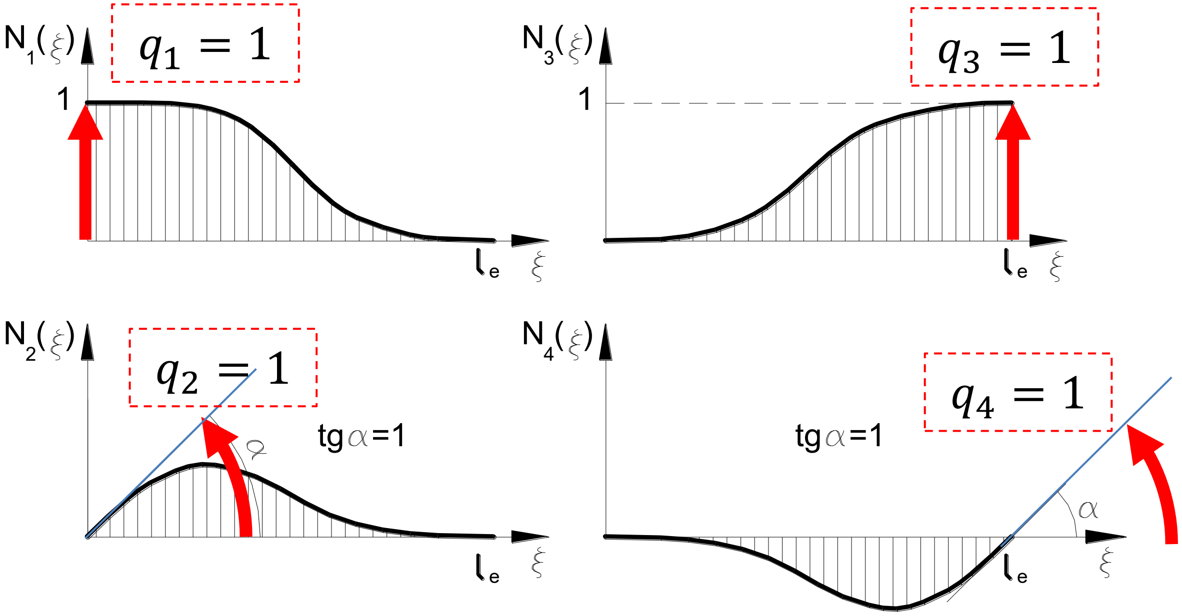
Shape functions of a beam element:

$$N_1(\xi)=1-3\frac{\xi^2}{l_e^2}+2\frac{\xi^3}{l_e^3},$$

$$N_2(\xi)=\xi-2\frac{\xi^2}{l_e}+\frac{\xi^3}{l_e^2},$$

$$N_3(\xi)=3\frac{\xi^2}{l_e^2}-2\frac{\xi^3}{l_e^3},$$

$$N_4(\xi)=\frac{-\xi^2}{l_e}+\frac{\xi^3}{l_e^2}.$$



A beam finite element - shape functions and their derivatives

$$N' = \frac{dN}{d\xi}, \quad N'' = \frac{d^2N}{d\xi^2}, \quad N''' = \frac{d^3N}{d\xi^3}$$

For the first shape function:

$$N_1' = -\frac{6}{l_e^2}\xi + \frac{6}{l_e^3}\xi^2, \quad N_1'' = -\frac{6}{l_e^2} + \frac{12}{l_e^3}\xi, \quad N_1''' = \frac{12}{l_e^3}$$

For other shape functions:

$$N_2' = 1 - \frac{4}{l_e}\xi + \frac{3}{l_e^2}\xi^2, \quad N_2'' = -\frac{4}{l_e} + \frac{6}{l_e^2}\xi, \quad N_2''' = \frac{6}{l_e^2}$$

$$N_3' = \frac{6}{l_e^2}\xi - \frac{6}{l_e^3}\xi^2, \quad N_3'' = \frac{6}{l_e^2} - \frac{12}{l_e^3}\xi, \quad N_3''' = -\frac{12}{l_e^3}$$

$$N_4' = -\frac{2}{l_e}\xi + \frac{3}{l_e^2}\xi^2, \quad N_4'' = -\frac{2}{l_e} + \frac{6}{l_e^2}\xi, \quad N_4''' = \frac{6}{l_e^2}$$

A beam finite element – total potential energy

Deflection function and its derivatives:

$$
\boxed{\begin{aligned}
w(\xi) &= \lfloor N(\xi) \rfloor \{q\}_e \,, \\
w'(\xi) &= \lfloor N'(\xi) \rfloor \{q\}_e \,, \\
w''(\xi) &= \lfloor N''(\xi) \rfloor \{q\}_e \,.
\end{aligned}}
$$

Total potential energy of a beam of length *l*$_e$:

$$V_e = U_e - W_{ze} = \frac{EI}{2}\int_0^{l_e} (w''(\xi))^2 d\xi - \int_0^{l_e} p(\xi) w(\xi) d\,\xi - \sum_i P_i w_i - \sum_j M_j \vartheta_j$$

$$U_e = \frac{EI}{2}\int_0^{l_e} w''(\xi) w''(\xi) d\xi = \frac{EI}{2}\int_0^{l_e} \lfloor q \rfloor_e \{N''\} \lfloor N'' \rfloor \{q\}_e d\xi =$$

$$= \frac{EI}{2} \lfloor q \rfloor_e \int_0^{l_e} \begin{bmatrix} N_1{}''N_1{}'' & N_1{}''N_2{}'' & N_1{}''N_3{}'' & N_1{}''N_4{}'' \\ N_2{}''N_1{}'' & N_2{}''N_2{}'' & N_2{}''N_3{}'' & N_2{}''N_4{}'' \\ N_3{}''N_1{}'' & N_3{}''N_2{}'' & N_3{}''N_3{}'' & N_3{}''N_4{}'' \\ N_4{}''N_1{}'' & N_4{}''N_2{}'' & N_4{}''N_3{}'' & N_4{}''N_4{}'' \end{bmatrix} d\xi \{q\}_e.$$

A beam finite element – stiffness matrix

Elastic strain energy of the beam:

$$U_e = \frac{1}{2}\lfloor q \rfloor_e [k]_e \{q\}_e$$

$$[k]_e = EI\begin{bmatrix} \int_0^{l_e} N_1''N_1''d\xi & \int_0^{l_e} N_1''N_2''d\xi & \int_0^{l_e} N_1''N_3''d\xi & \int_0^{l_e} N_1''N_4''d\xi \\ \int_0^{l_e} N_2''N_1''d\xi & \int_0^{l_e} N_2''N_2''d\xi & \int_0^{l_e} N_2''N_3''d\xi & \int_0^{l_e} N_2''N_4''d\xi \\ \int_0^{l_e} N_3''N_1''d\xi & \int_0^{l_e} N_3''N_2''d\xi & \int_0^{l_e} N_3''N_3''d\xi & \int_0^{l_e} N_3''N_4''d\xi \\ \int_0^{l_e} N_4''N_1''d\xi & \int_0^{l_e} N_4''N_2''d\xi & \int_0^{l_e} N_4''N_3''d\xi & \int_0^{l_e} N_4''N_4''d\xi \end{bmatrix}$$

Stiffness matrix of a beam element:

$$[k]_e = \frac{2EI}{l_e^3}\begin{bmatrix} 6 & 3l_e & -6 & 3l_e \\ 3l_e & 2l_e^2 & -3l_e & l_e^2 \\ -6 & -3l_e & 6 & -3l_e \\ 3l_e & l_e^2 & -3l_e & 2l_e^2 \end{bmatrix}$$

A beam finite element – equivalent forces

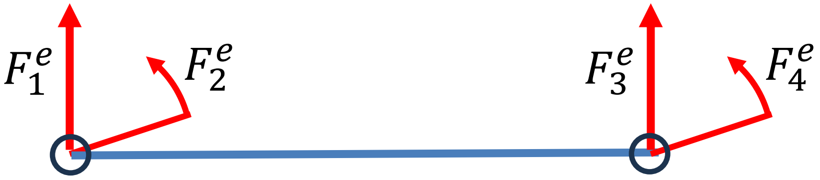
Work of external load: $$W_{ze}^{p} = \int_{0}^{l_e} p(\xi) w(\xi) d\xi = \int_{0}^{l_e} p(\xi) \lfloor N(\xi) \rfloor \{q\}_e \, d\xi$$

$$= \int_{0}^{l_e} \lfloor N_1(\xi) p(\xi),\ N_2(\xi) p(\xi),\ N_3(\xi) p(\xi),\ N_4(\xi) p(\xi) \rfloor \{q\}_e \, d\xi,$$

$$W_{ze}^{p} = \lfloor F_1^e, F_2^e, F_3^e, F_4^e \rfloor_e \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \lfloor F \rfloor_e \{q\}_e$$

Nodal equivalent forces resulting from continuous load output:

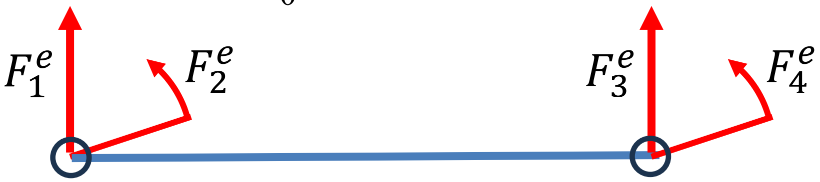
$$F_i^e = \int_{0}^{l_e} N_i(\xi) p(\xi) d\xi$$



Example: Equivalent forces resulting from uniform transverse loading

Nodal forces resulting from continuous loading:

$$F_i^e = \int_0^{l_e} N_i(\xi)\, p(\xi)\, d\xi$$



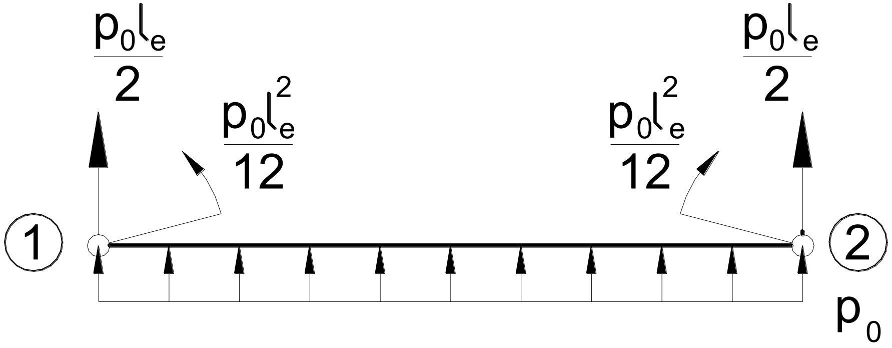
For uniform transverse loading:

$$F_1^e = \int_0^{l_e} N_1(\xi)\cdot p_0 \cdot d\xi = \int_0^{l_e} \left(1 - \frac{3}{l_e^2}\xi^2 + \frac{2}{l_e^3}\xi^3\right) p_0 \cdot d\xi = \frac{p_0 l_e}{2}$$

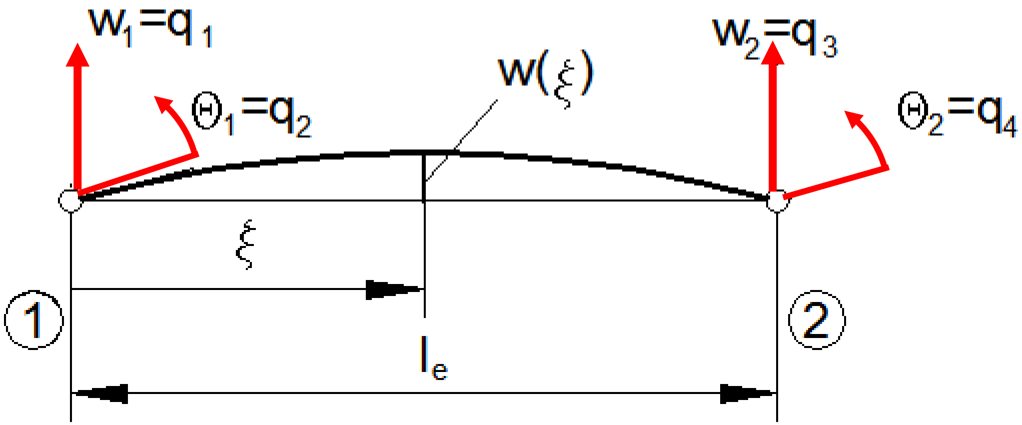
$$F_2^e = \int_0^{l_e} N_2(\xi)\cdot p_0\, d\xi = \int_0^{l_e} \left(\xi - \frac{2}{l_e}\xi^2 + \frac{1}{l_e^2}\xi^3\right) p_0 \cdot d\xi = \frac{p_0 l_e^2}{12}$$

e.t.c.

$$\boxed{\begin{aligned} F_1^e &= F_3^e = \frac{p_0 l_e}{2} \\ F_2^e &= \frac{p_0 l_e^2}{12} \\ F_4^e &= \frac{-p_0 l_e^2}{12} \end{aligned}}$$



A beam finite element – list of search functions



Vector of nodal parameters

$$\{q\}_e = \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}_e = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_e$$

Deflection:

$$w(\xi) = \underset{1\times 4}{\lfloor N \rfloor} \cdot \underset{4\times 1}{\{q\}_e}$$

– **Polynomial of the 3rd order**

Bending moment:

$$M_y(\xi) = EJ_y\, w'' = EJ_y \underset{1\times 4}{\lfloor N'' \rfloor} \cdot \underset{4\times 1}{\{q\}_e}$$

– **Linear function**

Shear force:

$$T_z(\xi) = -EJ_y\, w''' = -EJ_y \underset{1\times 4}{\lfloor N''' \rfloor} \cdot \underset{4\times 1}{\{q\}_e}$$

– **Constant value**

DOF Solution

$$\underset{N\times 1}{\{q\}} = \underset{N\times N}{[K]}^{-1} \cdot \underset{N\times 1}{\{F\}}$$

Element solution

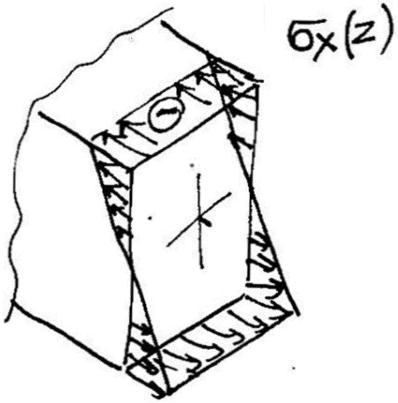
$$M_y(\xi) = EJ_y \underset{1\times 4}{\lfloor N''(\xi) \rfloor} \cdot \underset{4\times 1}{\{q\}_e} \; , \; T_z(\xi) = -EJ_y \underset{1\times 4}{\lfloor N'''(\xi) \rfloor} \cdot \underset{4\times 1}{\{q\}_e}$$

$$\sigma_x(\xi,z) = -M_y(\xi) \cdot \frac{z}{J_y} = -E \underset{1\times 4}{\lfloor N''(\xi) \rfloor} \cdot \underset{4\times 1}{\{q\}_e} \cdot z$$

$$\tau_{xz}(\xi,z) = f(z) \cdot T_z(\xi) \overset{\text{rectangle}}{\underset{b\times h}{=}} \frac{3}{2}\left(1-\left(\frac{2z}{h}\right)^2\right)/bh \cdot T_z(\xi)$$

$$\varepsilon_x(\xi,z) = \sigma_x(\xi,z)/E = -\underset{1\times 4}{\lfloor N'' \rfloor} \cdot \underset{4\times 1}{\{q\}_e} \cdot z$$

$$w(\xi) = \underset{1\times 4}{\lfloor N \rfloor} \cdot \underset{4\times 1}{\{q\}_e}$$



A beam finite element – system of equations

Total potential energy of the beam element:

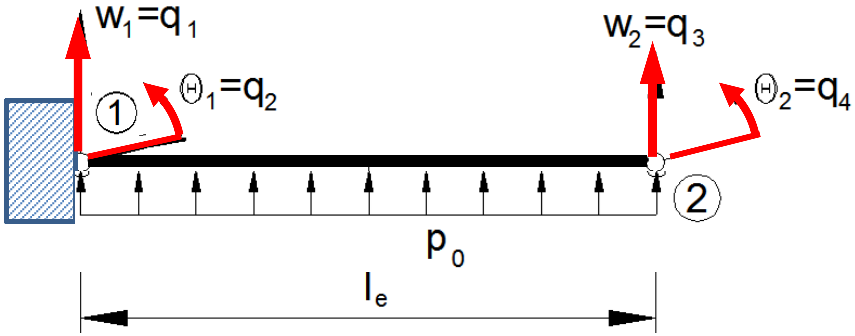
$$V_e = U_e - W_{ze} = \frac{1}{2}\underset{1\times 4}{\lfloor q \rfloor_e}\underset{4\times 4}{[k]_e}\underset{4\times 1}{\{q\}_e} - \underset{1\times 4}{\lfloor q \rfloor_e}\underset{4\times 1}{\{F\}_e}$$

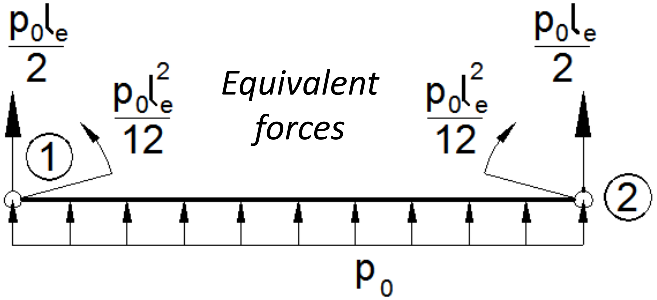
Condition for minimizing total potential energy:

$$\frac{\partial V_e}{\partial q_i} = 0 \qquad i = 1, 2, 3, \dots, n$$

$$\boxed{[k]_e \{q\}_e = \{F\}_e} \Rightarrow \frac{2EI}{l_e^3}\left[\begin{array}{|c|c|c|c|}\hline 6 & 3l_e & -6 & 3l_e \\ \hline 3l_e & 2l_e^2 & -3l_e & l_e^2 \\ \hline -6 & -3l_e & 6 & -3l_e \\ \hline 3l_e & l_e^2 & -3l_e & 2l_e^2 \\ \hline \end{array}\right] \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_e = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}_e$$

Example: cantilever beam loaded with a uniformly distributed transverse loading *(one element)*





Vector of nodal parameters:

$$\{q\}_e = \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}_e = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_e = \begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \end{Bmatrix}_e$$

$$\frac{2EI}{l^3} \begin{bmatrix} 6 & 3l & -6 & 3l \\ 3l & 2l^2 & -3l & l^2 \\ -6 & -3l & 6 & -3l \\ 3l & l^2 & -3l & 2l^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ \frac{p_0 l}{2} \\ \frac{-p_0 l^2}{12} \end{Bmatrix}$$

$$\frac{2EI}{l^3}(6q_3 - 3lq_4) = \frac{p_0 l}{2},$$
$$\frac{2EI}{l^3}(-3lq_3 + 2l^2 q_4) = \frac{-p_0 l^2}{12},$$

$$q_3 = \frac{1}{8}\frac{p_0 l^4}{EI} \qquad q_4 = \frac{1}{6}\frac{p_0 l^3}{EI}$$

$$w(\xi) = \sum_{i=1}^{4} N_i(\xi) q_i$$

$$w(\xi) = \left(\frac{3}{8} - \frac{1}{6}\right)\frac{p_0 l^2}{EI}\xi^2 + \left(\frac{-2}{8} + \frac{1}{6}\right)\frac{p_0 l}{EI}\xi^3 = \frac{5}{24}\frac{p_0 l^2}{EI}\xi^2 - \frac{p_0 l}{12EI}\xi^3$$

Example: cantilever beam loaded with a uniformly distributed transverse loading *(one element)*

Reactions:

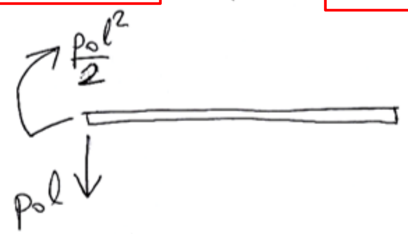
$$\frac{2EI}{l^3}\begin{bmatrix}6&3l&-6&3l\\3l&2l^2&-3l&l^2\\-6&-3l&6&-3l\\3l&l^2&-3l&2l^2\end{bmatrix}\begin{Bmatrix}0\\0\\q_3\\q_4\end{Bmatrix}=\begin{Bmatrix}F_1\\F_2\\\frac{p_0l}{2}\\-\frac{p_0l^2}{12}\end{Bmatrix}$$

$q_3=\frac{1}{8}\frac{p_0l^4}{EI}$

$q_4=\frac{1}{6}\frac{p_0l^3}{EI}$

$$\begin{cases}\frac{2EJ}{l^3}\left(-6\cdot q_3+3l\cdot q_4\right)=R_1+\frac{p_0l}{2}\\[2ex]\frac{2EJ}{l^3}\left(-3l\cdot q_3+l^2\cdot q_4\right)=R_2+\frac{p_0l^2}{12}\end{cases}$$

$\boxed{R_1=-p_0l}$; $\boxed{R_2=-\frac{p_0l^2}{2}}$

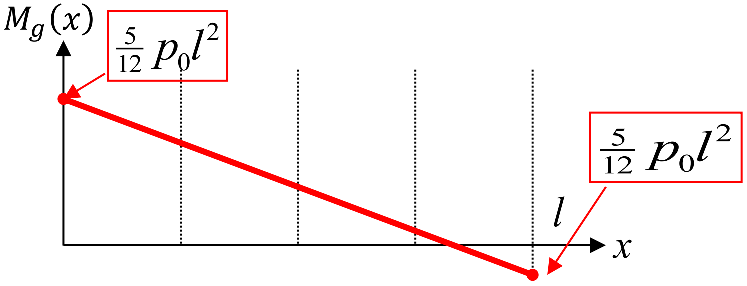


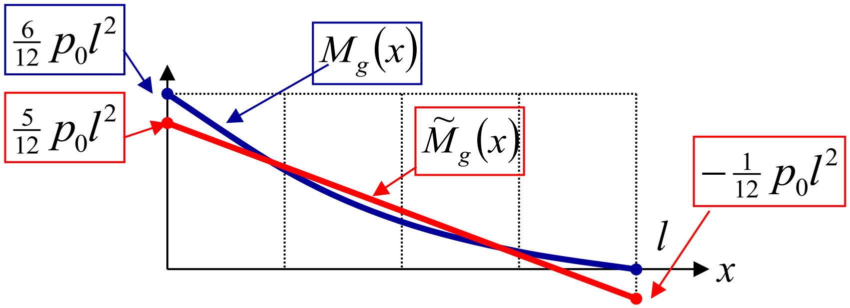
Bending moment:

$$M_g=EJ\cdot w''(\xi)=EJ\cdot\sum_{1}^{4}N_i''(\xi)\cdot q_i$$

$$M_g=EJ\left(N_3''(\xi)\cdot q_3+N_4''(\xi)\cdot q_4\right)$$

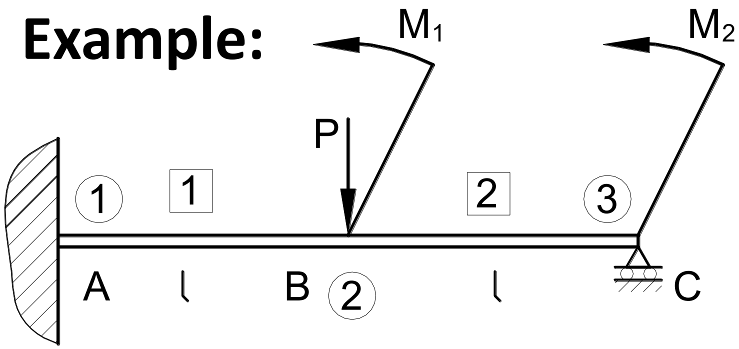
$$\boxed{M_g=\frac{5}{12}p_0l^2-\frac{p_0l}{2}\cdot\xi}$$





As in the Ritz solution!

Example:



$$\{q\}=\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{Bmatrix}=\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}=\begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ q_3 \\ q_4 \\ \mathbf{0} \\ q_6 \end{Bmatrix}$$

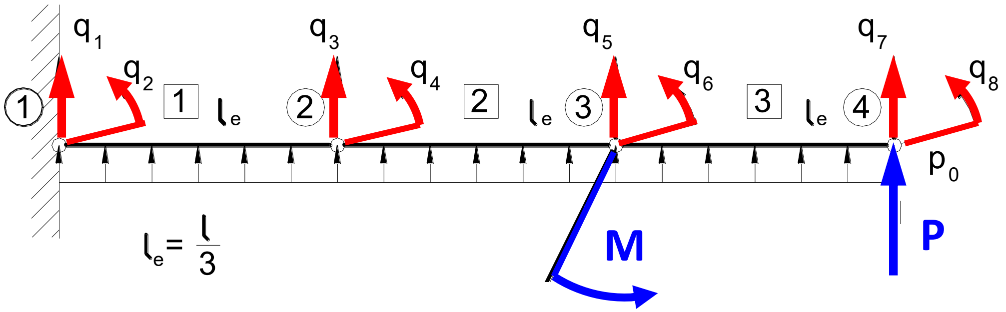
$$\{F\}=\begin{Bmatrix} \boldsymbol{F_1} \\ \boldsymbol{F_2} \\ -P \\ M_1 \\ \boldsymbol{F_5} \\ M_2 \end{Bmatrix}$$

$$[K]=\frac{2EI}{l^3}\begin{bmatrix} 6 & 3l & -6 & 3l & & \\ 3l & 2l^2 & -3l & l^2 & & \\ -6 & -3l & 6+6 & -3l+3l & -6 & 3l \\ 3l & l^2 & -3l+3l & 2l^2+2l^2 & -3l & l^2 \\ & & -6 & -3l & 6 & -3l \\ & & 3l & l^2 & -3l & 2l^2 \end{bmatrix}$$

$$\frac{2EI}{l^3}\begin{bmatrix} 12 & 0 & 3l \\ 0 & 4l^2 & l^2 \\ 3l & l^2 & 2l^2 \end{bmatrix}\begin{Bmatrix} q_3 \\ q_4 \\ q_6 \end{Bmatrix}=\begin{Bmatrix} -P \\ M_1 \\ M_2 \end{Bmatrix}$$

$$\begin{Bmatrix} q_3 \\ q_4 \\ q_6 \end{Bmatrix}=\begin{Bmatrix} w_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix}=\frac{l}{96EI}\begin{bmatrix} 7l^2 & 3l & -12l \\ 3l & 15 & -12 \\ -12l & -12 & 48 \end{bmatrix}\begin{Bmatrix} -P \\ M_1 \\ M_2 \end{Bmatrix}$$

Example: cantilever beam loaded with a uniformly distributed transverse loading and point loads (*three elements*)



Vector of nodal parameters:

$$\{q\} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix} = \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \\ w_4 \\ \theta_4 \end{Bmatrix}$$

Elastic strain energy in every element:

$$U_e = \frac{1}{2}\lfloor q \rfloor_e [k]_e \{q\}_e = \frac{1}{2}\underset{1\times N}{\lfloor q \rfloor}\underset{N\times N}{[k^*]}_e \underset{N\times 1}{\{q\}}$$

Extended Element Stiffness Matrices:

$[k^*]_1 =$

▨	▨	▨	▨				
▨	▨	▨	▨				
▨	▨	▨	▨				
▨	▨	▨	▨				

q_1, q_2, q_3, q_4

$[k^*]_2 =$

		▨	▨	▨	▨		
		▨	▨	▨	▨		
		▨	▨	▨	▨		
		▨	▨	▨	▨		

q_3, q_4, q_5, q_6

$[k^*]_3 =$

				▨	▨	▨	▨
				▨	▨	▨	▨
				▨	▨	▨	▨
				▨	▨	▨	▨

q_5, q_6, q_7, q_8

Example: cantilever beam loaded with a uniformly distributed transverse loading and point loads

(*three elements*)

Elastic strain energy of the entire beam:

$$U=\sum_{e=1}^{LE}U_e=\frac{1}{2}\lfloor q\rfloor\left(\sum_{i=1}^{LE}\left[k^*\right]_e\right)\{q\}=\frac{1}{2}\lfloor q\rfloor[K]\{q\}$$

Total potential energy of the system:

$$V=U-W_z=\frac{1}{2}\lfloor q\rfloor[K]\{q\}-\lfloor q\rfloor\{F\}$$

The condition of minimum total potential energy of the system:

$$\frac{\partial V}{\partial q_i}=0 \qquad i=1,2,3,\ldots,n$$

$$[K]\{q\}=\{F\}$$

+ displacement boundary conditions

$$M_q(\xi)=EIw''(\xi)=EI\left\lfloor N_1'',N_2'',N_3'',N_4''\right\rfloor\begin{Bmatrix}q_1\\q_2\\q_3\\q_4\end{Bmatrix}_e, \qquad M_q(\xi)=\left[\frac{12}{l_e^3}(\xi-\frac{l_e}{2})q_1+\frac{6}{l_e^2}(\xi-\frac{2}{3}l_e)q_2-\frac{12}{l_e^3}(\xi-\frac{l_e}{2})q_3+\frac{6}{l_e^2}(\xi-\frac{l_e}{3})q_4\right]EI,$$

$$T(\xi)=-EIw'''(\xi)=EI\left\lfloor N_1''',N_2''',N_3''',N_4'''\right\rfloor\begin{Bmatrix}q_1\\q_2\\q_3\\q_4\end{Bmatrix}_e. \qquad T(\xi)=-\left[\frac{12}{l_e^3}(q_1-q_3)+\frac{6}{l_e^2}(q_2+q_4)\right]EI.$$

Example: cantilever beam loaded with a uniformly distributed transverse loading and point loads

(three elements)

$$\begin{bmatrix}
k_{11}^{1} & k_{12}^{1} & k_{13}^{1} & k_{14}^{1} & 0 & 0 & 0 & 0 \\
k_{21}^{1} & k_{22}^{1} & k_{23}^{1} & k_{24}^{1} & 0 & 0 & 0 & 0 \\
k_{31}^{1} & k_{32}^{1} & k_{33}^{1}+k_{11}^{2} & k_{34}^{1}+k_{12}^{2} & k_{13}^{2} & k_{14}^{2} & 0 & 0 \\
k_{41}^{1} & k_{42}^{1} & k_{43}^{1}+k_{21}^{2} & k_{44}^{1}+k_{22}^{2} & k_{23}^{2} & k_{24}^{2} & 0 & 0 \\
0 & 0 & k_{31}^{2} & k_{32}^{2} & k_{33}^{2}+k_{11}^{3} & k_{34}^{2}+k_{12}^{3} & k_{13}^{3} & k_{14}^{3} \\
0 & 0 & k_{41}^{2} & k_{42}^{2} & k_{43}^{2}+k_{21}^{3} & k_{44}^{2}+k_{22}^{3} & k_{23}^{3} & k_{24}^{3} \\
0 & 0 & 0 & 0 & k_{31}^{3} & k_{32}^{3} & k_{33}^{3} & k_{34}^{3} \\
0 & 0 & 0 & 0 & k_{41}^{3} & k_{42}^{3} & k_{43}^{3} & k_{44}^{3}
\end{bmatrix}
\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix}
=
\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix}$$

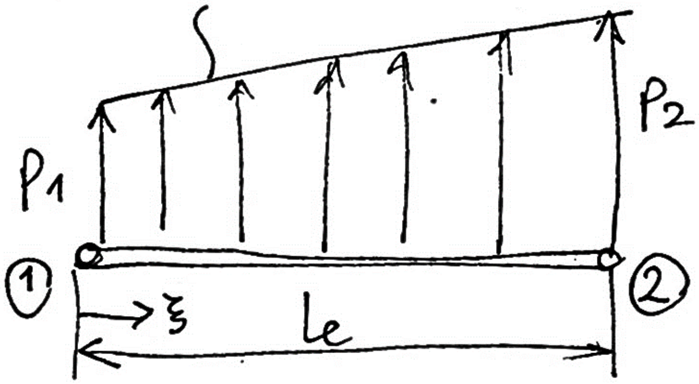
$$\frac{2EI}{l_e^3}
\begin{bmatrix}
6 & 3l_e & -6 & 3l_e & 0 & 0 & 0 & 0 \\
3l_e & 2l_e^2 & -3l_e & l_e^2 & 0 & 0 & 0 & 0 \\
-6 & -3l_e & 12 & 0 & -6 & 3l_e & 0 & 0 \\
3l_e & l_e^2 & 0 & 4l_e^2 & -3l_e & l_e^2 & 0 & 0 \\
0 & 0 & -6 & -3l_e & 12 & 0 & -6 & 3l_e \\
0 & 0 & 3l_e & l_e^2 & 0 & 4l_e^2 & -3l_e & l_e^2 \\
0 & 0 & 0 & 0 & -6 & -3l_e & 6 & -3l_e \\
0 & 0 & 0 & 0 & 3l_e & l_e^2 & -3l_e & 2l_e^2
\end{bmatrix}
\begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix}
=
\begin{Bmatrix} F_1 \\ F_2 \\ p_0 l_e \\ 0 \\ p_0 l_e \\ M \\ P + \dfrac{p_0 l_e}{2} \\ \dfrac{-p_0 l_e^2}{12} \end{Bmatrix}$$

Typical FEM calculations

1. Determination of the stiffness matrix of the elements $[k]_e$
2. Aggregation of the matrix of elements into the global matrix $[K]$
3. Determination of the equivalent load vector $\{F\}$
4. Introduction of boundary conditions – determination of all the searched parameters $\{q\}$
5. Determination of internal forces (moments and shear forces) and normal and shear stresses

Example: find the components of the equivalent load for a linearly distributed transverse load

$$p(\xi) = \frac{p_2 - p_1}{l_e} \cdot \xi + p_1$$



$$\lfloor F \rfloor_e = \lfloor F_{1e}, F_{2e}, F_{3e}, F_{4e} \rfloor_e$$

1) Transverse equivalent force at node 1:

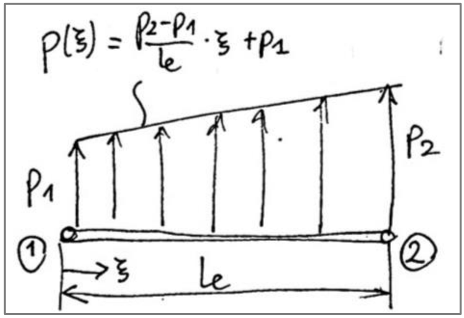
$$F_{1e} = \int_0^{l_e} p(\xi) \cdot N_1(\xi)\, d\xi = \int_0^{l_e} \left(\frac{p_2 - p_1}{l_e} \cdot \xi + p_1 \right) \left(1 - \frac{3}{l_e^2} \xi^2 + \frac{2}{l_e^3} \xi^3 \right) d\xi =$$

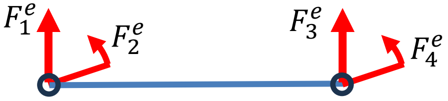
$$= \boxed{\frac{p_1 l_e}{2} + \frac{3}{20} (p_2 - p_1) \cdot l_e}$$

2) Equivalent moment at node 1:

$$F_{2e} = \int_0^{l_e} p(\xi)\cdot N_2(\xi)\,d\xi = \int_0^{l_e} \left(\frac{p_2-p_1}{l_e}\cdot\xi + p_1\right)\left(\xi - \frac{2}{l_e}\xi^2 + \frac{1}{l_e^2}\xi^3\right)d\xi =$$

$$= \boxed{\frac{p_1 l_e^2}{12} + \frac{1}{30}(p_2-p_1)\,l_e^2}$$





3) Transverse equivalent force at node 2:

$$F_{3e} = \int_0^{l_e} p(\xi)\,N_3(\xi)\,d\xi = \int_0^{l_e} \left(\frac{p_2-p_1}{l_e}\cdot\xi + p_1\right)\left(\frac{3}{l_e^2}\xi^2 - \frac{2}{l_e^3}\xi^3\right)d\xi =$$

$$= \boxed{\frac{p_1 l_e}{2} + \frac{7}{20}(p_2-p_1)\,l_e}$$

4) Equivalent moment at node 2:

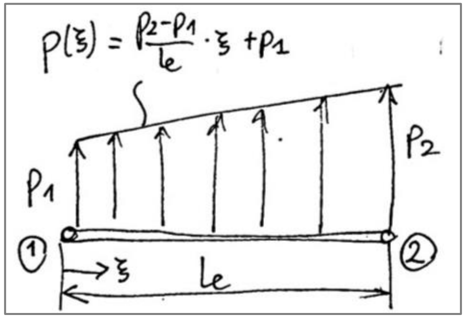
$$F_{4e} = \int_0^{l_e} p(\xi)\cdot N_4(\xi)\,d\xi = \int_0^{l_e} \left(\frac{p_2-p_1}{l_e}\cdot\xi + p_1\right)\left(-\frac{1}{l_e}\xi^2 + \frac{1}{l_e^2}\xi^3\right)d\xi =$$

$$= \boxed{-\frac{p_1 l_e^2}{12} - \frac{(p_2-p_1)\,l_e^2}{20}}$$

Example: find the components of the equivalent load for a linearly distributed transverse load

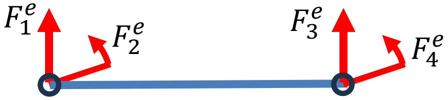
$$F_{1e} = \boxed{\frac{p_1 l_e}{2} + \frac{3}{20}(p_2-p_1)\cdot l_e}$$

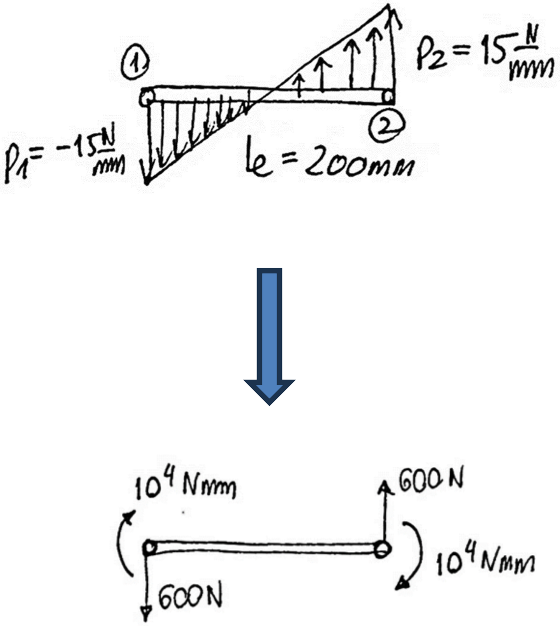
$$F_{2e} = \boxed{\frac{p_1 l_e^2}{12} + \frac{1}{30}(p_2-p_1) l_e^2}$$



$$F_{3e} = \boxed{\frac{p_1 l_e}{2} + \frac{7}{20}(p_2-p_1) l_e}$$

$$F_{4e} = \boxed{-\frac{p_1 l_e^2}{12} - \frac{(p_2-p_1) l_e^2}{20}}$$





$$F_{1e} = \frac{-15\frac{N}{mm}\cdot 200\,mm}{2} + \frac{3}{20}\left(15\frac{N}{mm} - \left(-15\frac{N}{mm}\right)\right)\cdot 200mm = -600\,N$$

$$F_{2e} = \frac{-15\frac{N}{mm}\cdot 200^2 mm^2}{12} + \frac{1}{30}\left(30\frac{N}{mm}\right)\cdot 200^2 mm^2 = -10^4\,Nmm$$

$$F_{3e} = \frac{-15\frac{N}{mm}\cdot 200\,mm}{2} + \frac{7}{20}\left(30\frac{N}{mm}\right)\cdot 200\,mm = 600\,N$$

$$F_{4e} = -\frac{\left(-15\frac{N}{mm}\right)\cdot 200^2 mm^2}{12} - \frac{\left(30\frac{N}{mm}\right)}{20}\cdot 200^2 mm^2 = -10^4\,Nmm$$